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# GEOMETRY OF MANIFOLDS WITH STRUCTURAL GROUP $\mathcal{U}(n) \times \mathcal{O}(s)$

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K. Yano [12], [13] has introduced the notion of an $f$-structure on a $C^\infty$ manifold $M^{2n+s}$, i.e., a tensor field $f$ of type (1, 1) and rank $2n$ satisfying $f^3 + f = 0$, the existence of which is equivalent to a reduction of the structural group of the tangent bundle to $\mathcal{U}(n) \times \mathcal{O}(s)$. Almost complex ($s = 0$) and almost contact ($s = 1$) structures are well-known examples of $f$-structures. An $f$-structure with $s = 2$ has arisen in the study of hypersurfaces in almost contact spaces [3]; this structure has been studied further by S. I. Goldberg and K. Yano [4].

The purpose of the present paper is to introduce for manifolds with an $f$-structure the analogue of the Kaehler structure in the almost complex case and of the quasi-Sasakian structure [2] in the almost contact case, and to begin the study of the geometry of manifolds with such a structure. In § 1 we introduce the Kaehler anologue and its geometry and in § 2 we study $f$-sectional curvature. § 3 discusses principal toroidal bundles and § 4 generalizes the Hopf-fibration to give a canonical example of a manifold with an $f$-structure playing the role of complex projective space in Kaehler geometry and the odd-dimensional sphere in Sasakian geometry.

**1.** Let $M^{2n+s}$ be a manifold with an $f$-structure of rank $2n$. If there exists on $M^{2n+s}$ vector fields $\xi_x, x = 1, \cdots, s$ such that if $\eta_x$ are dual 1-forms, then

$$(*) \qquad \begin{aligned} \eta_x(\xi_y) &= \delta_{xy} , \\ f\xi_x = 0 , &\quad \eta_x \circ f = 0 , \\ f^2 &= -I + \sum \xi_x \otimes \eta_x , \end{aligned}$$

we say that the $f$-structure has *complemented frames*. If $M^{2n+s}$ has an $f$-structure with complemented frames, then there exists on $M^{2n+s}$ a Riemannian metric $g$ such that

$$g(X, Y) = g(fX, fY) + \sum \eta_x(X)\eta_x(Y) ,$$

where $X, Y$ are vector fields on $M^{2n+s}$ [13], and we say $M^{2n+s}$ has a *metric $f$-structure*. Define the *fundamental 2-form* $F$ by

$$F(X, Y) = g(X, fY) .$$

Further we say an $f$-structure is *normal* if it has complemented frames and

$$[f, f] + \sum \xi_x \otimes d\eta_x = 0 ,$$

where $[f, f]$ is the Nijenhuis torsion of $f$ [9]. Finally a metric $f$-structure which is normal and has closed fundamental 2-form will be called a $\mathscr{K}$*-structure* and $M^{2n+s}$ a $\mathscr{K}$*-manifold,*

It should be noted that since $\eta_1 \wedge \cdots \wedge \eta_s \wedge F^n \neq 0$, a $\mathscr{K}$-manifold is orientable.

Two cases will be of special interest.

1) Let $M^{2n+s}$ be a Riemannian manifold with global linearly independent 1-forms $\eta_1, \cdots, \eta_s$ such that $d\eta_1 = \cdots = d\eta_s$ and

$$\eta_1 \wedge \cdots \wedge \eta_s \wedge (d\eta_x)^n \neq 0 .$$

Let $\mathscr{L}(m) = \{X \in M_m^{2n+s}, m \in M^{2n+s} \mid \eta_x(X) = 0, x = 1, \cdots, s\}$; then $\mathscr{L}$ determines a distribution which together with its complement reduces the structural group to $\mathscr{O}(2n) \times \mathscr{O}(s)$. Now if $\xi_1, \cdots, \xi_s$ are vector fields dual to $\eta_1, \cdots, \eta_s$ and $X_1, \cdots, X_{2n}$ linearly independent vector fields in $\mathscr{L}$, then

$$\begin{aligned}&(\eta_1 \wedge \cdots \wedge \eta_s \wedge (d\eta_x)^n)(\xi_1, \cdots, \xi_s, X_1, \cdots, X_{2n})\\ &= (d\eta_x)^n(X_1, \cdots, X_{2n}) \neq 0\end{aligned}$$

giving $\mathscr{L}$ a symplectic structure. Thus the structural group can be reduced to $\mathscr{U}(n) \times \mathscr{O}(s)$ and $M^{2n+s}$ has a metric $f$-structure with complemented frames $\eta_1, \cdots, \eta_s$ and fundamental 2-form $F = d\eta_x$. If this structure is a $\mathscr{K}$-structure, we will call it an $\mathscr{S}$*-structure*.

2) Let $M^{2n+s}$ be a manifold with a $\mathscr{K}$-structure with $\eta_1, \cdots, \eta_s$ denoting the complemented frames. If $d\eta_x = 0, x = 1, \cdots, s$, we call it a $\mathscr{C}$*-structure.*

**Theorem 1.1.** *On a $\mathscr{K}$-manifold the vector fields $\xi_1, \cdots, \xi_s$ are Killing.*

*Proof.* Denoting Lie differentiation by $\mathscr{L}$ we

$$\begin{aligned}(\mathscr{L}_{\xi_x} F)(X, Y) &= \xi_x F(X, Y) - F([\xi_x, X], Y) - F(X, [\xi_x, Y])\\ &= \xi_x g(X, fY) - g([\xi_x, X], fY) - g(X, [\xi_x, fY])\\ &= (\mathscr{L}_{\xi_x} g)(X, fY) ,\end{aligned}$$

where we have used the fact that $\mathscr{L}_{\xi_x} f = 0$ (see [9]). But $\mathscr{L}_{\xi_x} F = d i_{\xi_x} F + i_{\xi_x} dF = 0$ since $(i_{\xi_x} F)X = F(\xi_x, X) = 0$. On the other hand,

$$\begin{aligned}(\mathscr{L}_{\xi_x} g)(X, \eta_y(Y)\xi_y) &= \xi_x(\eta_y(Y)\eta_y(X)) - \eta_y(Y)\eta_y([\xi_x, X])\\ &\quad - \eta_y(Y) g(X, [\xi_x, \xi_y]) - \xi_x(\eta_y(Y))\eta_y(X)\\ &= \eta_y(Y)\xi_x\eta_y(X) - \eta_y(Y)\eta_y([\xi_x, X])\\ &\quad - \eta_y(Y) g(X, [\xi_x, \xi_y]) = 0 ,\end{aligned}$$

since $\mathscr{L}_{\xi_x}\eta_y = 0$ and $\mathscr{L}_{\xi_x}\xi_y = 0$ (see [9]). Therefore

$$(\mathscr{L}_{\xi_x}g)(X, fY + \sum \eta_y(Y)\xi_y) = 0 ,$$

but $f + \sum \xi_y \otimes \eta_y$ is non-singular, hence $\mathscr{L}_{\xi_x}g = 0$.

**Lemma 1.2.** *On a $\mathscr{K}$-manifold $d\eta_x(X, Y) = -2(\nabla_Y\eta_x)(X)$ where $\nabla$ denotes covariant differentiation with respect to the Riemannian connexion. In the case of an $\mathscr{S}$-structure*

$$\nabla_Y\xi_x = -\tfrac{1}{2}fY ,$$

*and in the case of a $\mathscr{C}$-structure*

$$\nabla_Y\xi_x = 0 .$$

*Proof.* $d\eta_x(X, Y) = (\nabla_X\eta_x)(Y) - (\nabla_Y\eta_x)(X) = -2(\nabla_Y\eta_x)(X)$ since $\eta_x$ is Killing. In the case of an $\mathscr{S}$-structure we have $F = d\eta_x$ and hence $g(X, fY) = -2g(X, \nabla_Y\xi_x)$, whereas in the case of a $\mathscr{C}$-structure $0 = d\eta_x(X, Y) = -2g(X, \nabla_Y\xi_x)$.

We now discuss the meaning of $\nabla_XF$ for $\mathscr{K}$-structures.

**Proposition 1.3.** *On a $\mathscr{K}$-manifold*

$$(\nabla_XF)(Y, Z) = \tfrac{1}{2}\sum (\eta_x(Y)d\eta_x(fZ, X) + \eta_x(Z)d\eta_x(X, fY)) .$$

The proof is a very lengthy computation but similar to that given by Sasaki and Hatakeyama [10] for a Sasakian manifold.

**Proposition 1.4.** *On an $\mathscr{S}$-manifold*

$$\begin{aligned}(\nabla_XF)(Y, Z) = {} & \tfrac{1}{2}\sum (\eta_x(Y)g(X, Z) - \eta_x(Z)g(X, Y)) \\ & - \tfrac{1}{2}\sum_{x,y} \eta_y(X)(\eta_x(Y)\eta_y(Z) - \eta_x(Z)\eta_y(Y)) .\end{aligned}$$

*Proof.* In this case $F = d\eta_x$, $x = 1, \cdots, s$, hence Proposition 1.3 becomes

$$\begin{aligned}(\nabla_XF)(Y, Z) & = \tfrac{1}{2}\sum (\eta_x(Y)g(fZ, fX) - \eta_x(Z)g(fX, fY)) \\ & = \tfrac{1}{2}\sum_x (\eta_x(Y)g(X, Z) - \eta_x(Y)\sum_y \eta_y(X)\eta_y(Z)) \\ & \quad - \tfrac{1}{2}\sum_x (\eta_x(Z)g(X, Y) - \eta_x(Z)\sum_y \eta_y(X)\eta_y(Y)) ,\end{aligned}$$

which except for arrangement of terms is the desired formula.

**Theorem 1.5.** *A $\mathscr{K}$-structure is a $\mathscr{C}$-structure if and only if $\nabla F = 0$.*

*Proof.* $\nabla F = 0$ implies $[f, f] = 0$ and hence by normality $\sum d\eta_x(X, Y)\xi_x = 0$, but $\xi_1, \cdots, \xi_s$ are linearly independent therefore $d\eta_x = 0, x = 1, \cdots, s$ giving us a $\mathscr{C}$-structure. Conversely if $d\eta_x = 0, x = 1, \cdots, s$, then by Proposition 1.3 it is clear that $\nabla F = 0$.

Let $\mathscr{L}$ denote the distribution determined by $-f^2$ and $\mathscr{M}$ the complement distribution; $\mathscr{M}$ is determined by $f^2 + I$ and spanned by $\xi_1, \cdots, \xi_s$. Let $p = 2f^2 + I$ be the difference of the projection maps $f^2 + I$ and $-f^2$.

**Theorem 1.6.** *A $\mathscr{C}$-manifold $M^{2n+s}$ is a locally decomposable Riemannian manifold which is locally the product of a Kaehler manifold $M_1^{2n}$ and an Abelian Lie group $M_2^s$.*

*Proof.* $\nabla_X f = 0$ implies $\nabla_X f^2 = 0$ and hence $\nabla_X p = 0$ which is the condition for $M^{2n+s}$ to be locally decomposable [14, p. 221] and in turn locally the product of Riemannian manifolds $M_1^{2n}$ and $M_2^s$. Now restricting $f, g$ to $M_1^{2n}$ and again denoting them by $f, g$ we have $f^2 = -I$ and $g(fX, fY) = g(X, Y)$. Further since $\nabla_X f = 0$ we have $[f, f] = 0$, and from $dF = 0$ on $M^{2n+s}$ we have on $M_1^{2n}$, $dF = 0, F^n \neq 0$ where $F$ also denotes the fundamental 2-form on $M_1^{2n}$. Thus $M_1^{2n}$ is Kaehlerian.

To show that $M_2^s$ is an Abelian Lie group we show that $M^{2n+s}$ is locally the product of $M_1^{2n}$ and $s$ 1-dimensional manifolds. The integrability condition for such a structure is $h = 0$ [11] where in our case

$$h = \tfrac{1}{2} \sum (\xi_x \otimes \eta_x)[\xi_x \otimes \eta_x, \xi_x \otimes \eta_x] - \tfrac{1}{2} f^2[f^2, f^2] .$$

Since $[f^2, f^2] = 0$, from $\nabla_X f^2 = 0$ we have

$$\begin{aligned} h(X, Y) = \tfrac{1}{2} \sum \eta_x(\eta_x([X, Y])\xi_x &+ [\eta_x(X)\xi_x, \eta_x(Y)\xi_x] \\ &- \eta_x([\eta_x(X)\xi_x, Y])\xi_x - \eta_x([X, \eta_x(Y)\xi_x])\xi_x)\xi_x . \end{aligned}$$

Now if $X, Y \in \mathscr{L}$, then $[X, Y] \in \mathscr{L}$ since the distribution $\mathscr{L}$ determined by $-f^2$ is integrable, and it is easy to see that $h(X, Y) = 0$. If $X, Y \in \mathscr{M}$ it suffices to take $X = \xi_y, Y = \xi_z$ since $\xi_1, \cdots, \xi_s$ can be taken as part of a basis, but $[\xi_y, \xi_z] = 0$ and $h(\xi_y, \xi_z) = 0$ follow easily. Finally if $X = \xi_y$ and $Y \in \mathscr{L}$, we have

$$h(\xi_y, Y) = \tfrac{1}{2} \sum_x (\eta_x([\xi_y, Y])\xi_x - \eta_x(\eta_x(\xi_y)[\xi_x, Y])\xi_x) ,$$

but from the coboundary formula $d\eta_x(X, Y) = X\eta_x(Y) - Y\eta_x(X) - \eta([X, Y])$ we have $\eta_x([\xi_y, Y]) = 0$; hence $h(\xi_y, Y) = 0$.

Theorems 1.5, 1.6 should be compared with the corresponding results for for cosymplectic manifolds ($s = 1$) [2].

We close this section with some results on the curvature of $\mathscr{K}$-manifolds.

**Theorem 1.7.** *In both the $\mathscr{S}$-structure and $\mathscr{C}$-structure cases the distribution $\mathscr{M}$ is flat, i.e., all sectional curvatures $K(X, Y)$ for sections spanned by $X, Y \in \mathscr{M}$ vanish. In the $\mathscr{S}$-structure case sectional curvatures $K(X, Y)$ with $X \in \mathscr{L}, Y = \xi_x$ have value $1/4$. In the $\mathscr{C}$-structure case sectional curvatures with $X \in \mathscr{L}, Y \in \mathscr{M}$ vanish.*

*Proof.* In the $\mathscr{S}$-structure case using Lemma 1.2 and $\mathscr{L}_{\xi_x} f = 0$ we have

$$\begin{aligned} R_{\xi_x X}\xi_y &= \nabla_{[\xi_x, X]}\xi_y + \nabla_X \nabla_{\xi_x}\xi_y - \nabla_{\xi_x}\nabla_X \xi_y \\ &= -\tfrac{1}{2}f[\xi_x, X] + \tfrac{1}{2}\nabla_{\xi_x} fX \\ &= -\tfrac{1}{2}f[\xi_x, X] + \tfrac{1}{2}\nabla_{fX}\xi_x + \tfrac{1}{2}[\xi_x, fX] \\ &= -\tfrac{1}{4}f^2 X = \begin{cases} \tfrac{1}{4}X, X \in \mathscr{L} , \\ 0, X \in \mathscr{M} , \end{cases} \end{aligned}$$

from which the results for this case follow. For the $\mathscr{C}$-structure case, $\nabla_Y \xi_x = 0$ for every $Y$ gives $R_{\xi_x X}\xi_y = 0$ immediately.

**Corollary 1.8.** *A $\mathscr{C}$-manifold $M^{2n+s}$, $s \geq 2$, of constant curvature is locally flat.*

**Corollary 1.9.** *There are no $\mathscr{S}$-manifolds $M^{2n+s}$, $s \geq 2$ of constant curvature of strictly positive curvature.*

These results should be compared with those in the cases of $s = 0, s = 1$ (see e.g. [1], [2], [5]).

**2.** A plane section is called an *f-section* if it is determined by a vector $X \in \mathscr{L}(m), m \in M^{2n+s}$ such that $\{X, fX\}$ is an orthonormal pair spanning the section. The sectional curvature $K(X, fX)$, denoted $H(X)$, is called an *f-sectional curvature*.

Define a tensor $P$ of type $(0, 4)$ as follows (cf. [8]):

$$\begin{aligned} P(X, Y; Z, W) = F(X, Z)g(Y, W) - F(X, W)g(Y, Z) \\ - F(Y, Z)g(X, W) + F(Y, W)g(X, Z) . \end{aligned}$$

The following properties of $P$ follow directly from the definition.

**Lemma 2.1.** a) $P(X, Y; Z, W) = -P(Z, W; X, Y)$. b) *Let* $\{X, Y\}, X, Y \in \mathscr{L}$, *be an orthonormal pair, and set* $g(X, fY) = \cos\theta$, $0 \leq \theta \leq \pi$. *Then* $P(X, Y; X, fY) = -\sin^2\theta$.

**Lemma 2.2.** *On an $\mathscr{S}$-manifold $M^{2n+s}$,*

a) $g(R_{XY}Z, fW) + g(R_{XY}fZ, W) = (s/4)P(X, Y; Z, W) + Q(X, Y; Z, W)$, *where*

$$\begin{aligned} Q(X, Y; Z, W) = &\tfrac{1}{4}g(W, fY)(s \textstyle\sum \eta_x(X)\eta_x(Z) - \sum_{x,y} \eta_x(Z)\eta_y(X)) \\ &- \tfrac{1}{4}g(W, fX)(s \textstyle\sum \eta_x(Y)\eta_x(Z) - \sum_{x,y} \eta_x(Z)\eta_y(Y)) \\ &- \tfrac{1}{4}g(Z, fY)(s \textstyle\sum \eta_x(X)\eta_x(W) - \sum_{x,y} \eta_x(W)\eta_y(X)) \\ &+ \tfrac{1}{4}g(Z, fX)(s \textstyle\sum \eta_x(Y)\eta_x(W) - \sum_{x,y} \eta_x(W)\eta_y(Y)) . \end{aligned}$$

*Also if $X, Y, Z, W \in \mathscr{L}$, then $Q(X, Y; Z, W) = 0$ and*

b) $g(R_{fXfY}fZ, fW) = g(R_{XY}Z, W)$,

c) $g(R_{XfX}Y, fY) = g(R_{XY}X, Y) + g(R_{XfY}X, fY) + (s/2)P(X, Y; X, fY)$,

d) $g(R_{fXY}fX, Y) = g(R_{XfY}X, fY)$.

*Proof.* A direct computation shows that

$$(\nabla_{[X,Y]}F + \nabla_Y\nabla_X F - \nabla_X\nabla_Y F)(Z, W) = -g(R_{XY}Z, fW) - g(R_{XY}fZ, W) .$$

On the other hand using Proposition 1.4 and Lemma 1.2 to compute this we obtain a). Using a) twice and equations (*) we obtain b). Writing $g(R_{XfX}Y, fY) = -g(R_{XY}fY, X) - g(R_{XfY}X, Y)$ c) follows from a) and Lemma 2.1. Finally applying a) twice and the definition of $P$ we get d).

**Lemma 2.3.** *On a $\mathscr{C}$-manifold* a) $g(R_{XY}Z, fW) + g(R_{XY}fZ, W) = 0$. *Also if $X, Y, Z, W \in \mathscr{L}$, then* b) $g(R_{fXfY}fZ, fW) = g(R_{XY}Z, W)$, c) $g(R_{XfX}Y, fY) = g(R_{XY}X, Y) + g(R_{XfY}X, fY)$, d) $g(R_{fXY}fX, Y) = g(R_{XfY}X, fY)$.

*Proof.* The proof is similar to that of Lemma 2.2 but in the case of a) is much easier due to Theorem 1.5

**Lemma 2.4.** *Let $B(X, Y) = g(R_{XY}X, Y)$ and for $X \in \mathscr{L}$, $D(X) = B(X, fX)$. On an $\mathscr{S}$-manifold for $X, Y \in \mathscr{L}$ we have*

$$B(X, Y) = \frac{1}{32}[3D(X + fY) + 3D(X - fY) - D(X + Y) - D(X - Y) - 4D(X) - 4D(Y) - 6sP(X, Y; X, fX)] .$$

*On a $\mathscr{C}$-manifold for $X, Y \in \mathscr{L}$ we have*

$$B(X, Y) = \frac{1}{32}[3D(X + fY) + 3D(X - fY) - D(X + Y) - D(X - Y) - 4D(X) - 4D(Y)] .$$

*Proof.* A direct expansion gives

$$\begin{aligned} &\frac{1}{32}[3D(X + fY) + 3D(X - fY) - D(X + Y) - D(X - Y) \\ &\qquad - 4D(X) - 4D(Y) - 6sP(X, Y; X, fY)] \\ &= \frac{1}{32}[6g(R_{XY}X, Y) + 6g(R_{fX,fY}fX, fY) + 8g(R_{XfX}Y, fY) \\ &\qquad + 12g((R_{XY}fX, fY) - 2g(R_{XfY}X, fY) - 2g(R_{fXY}fX, Y) \\ &\qquad + 4g(R_{XfY}fX, Y) - 6sP(X, Y; X, fY)] . \end{aligned}$$

Applying Lemma 2.2 this becomes

$$\begin{aligned} &\frac{1}{32}[6g(R_{XY}X, Y) + 6g(R_{XY}X, Y) + 8g(R_{XY}X, Y) + 8g(R_{XfY}X, fY) \\ &\qquad + 4sP(X, Y; X, fY) + 12g(R_{XY}X, Y) + 3sP(X, Y; X, fY) \\ &\qquad - 2g(R_{XfY}X, fY) - 2g(R_{XfY}X, fY) - 4g(R_{XfY}X, fY) \\ &\qquad + sP(X, fY; X, Y) - 6sP(X, Y; X, fY)] \\ &= g(R_{XY}X, Y) . \end{aligned}$$

The proof in the case of a $\mathscr{C}$-manifold is similar by using Lemma 2.3.

If now $\{X, Y\}$ is an orthonormal pair in $\mathscr{L}$ and $g(X, fY) = \cos\theta, 0 \leq \theta \leq \pi$, then $K(X, Y) = B(X, Y)$ and, by straightforward computation, $D(X) = H(X), D(Y) = H(Y), D(X + fY) = 4(1 + \cos\theta)^2 H(X + fY), D(X - fY) = 4(1 - \cos\theta)^2 H(X - fY), D(X + Y) = 4H(X + Y), D(X - Y) = 4H(X - Y)$. Using Lemma 2.1, Lemma 2.4 now becomes

**Proposition 2.5.** *On an $\mathscr{S}$-manifold for an orthonormal pair $\{X, Y\}$ in $\mathscr{L}$ we have*

$$K(X, Y) = \frac{1}{8}\Big[3(1 + \cos\theta)^2 H(X + fY) + 3(1 - \cos\theta)^2 H(X - fY) - H(X + Y) - H(X - Y) - H(X) - H(Y) + \frac{3s}{2}\sin^2\theta\Big].$$

*In the case of a $\mathscr{C}$-manifold the formula is the same except that the last term is not present.*

**Theorem 2.6.** *The f-sectional curvatures determine the curvature of an $\mathscr{S}$-manifold or a $\mathscr{C}$-manifold completely.*

*Proof.* In addition to Theorem 1.7 some other curvature formulas are needed. It follows easily from Theorem 1.7 that in both cases $R_{\xi_x\xi_y}X = 0$ for all $X$. In the $\mathscr{S}$-manifold case, if $X \in \mathscr{L}$ is a unit vector then $g(R_{X\xi_x}X, \xi_y) = g(R_{\xi_x X}\xi_y, X) = 1/4$ and hence $R_{X\xi_x}X = (1/4)\sum \xi_z + Y, Y \in \mathscr{L}$; but

$$\begin{aligned} g(R_{X\xi_x}X, Y) &= -g(R_{XY}f^2X, \xi_x) \\ &= g(R_{XY}fX, f\xi_x) - \frac{s}{4}P(X, Y; fX, \xi_x) \\ &\quad - Q(X, Y; fX, \xi_x) = 0, \end{aligned}$$

so that $R_{X\xi_x}X = (1/4)\sum \xi_z$. In the $\mathscr{C}$-manifold case $R_{X\xi_x}X$ is easily checked.

Now let $\{X, Y\}$ be orthonormal pair, and write $X = aZ + \sum \eta_x(X)\xi_X$, $Y = bW + \sum \eta_x(Y)\xi_x$ where $a^2 + \sum \eta_x(X)^2 = 1, b^2 + \sum \eta_x(Y)^2 = 1$ and $Z, W$ are unit vectors in $\mathscr{L}$. Then after using the above curvature formulas the lengthy expansion of $K(X, Y) = g(R_{XY}X, Y)$ yields

$$\begin{aligned} K(X, Y) &= \frac{b^2}{4}\Big(\sum_{x,y} \eta_x(X)\eta_y(X)\Big) + \frac{a^2}{4}\Big(\sum_{x,y} \eta_x(Y)\eta_y(Y)\Big) \\ &\quad + \frac{1}{2}\Big(\sum_{x,y} \eta_x(X)\eta_y(Y)\Big)\Big(\sum \eta_z(X)\eta_z(Y)\Big) \\ &\quad + (a^2b^2 - (\sum \eta_x(X)\eta_x(Y))^2)K(Z, W) \end{aligned}$$

in the $\mathscr{S}$-manifold case and

$$K(X, Y) = (a^2b^2 - (\sum \eta_x(X)\eta_x(Y))^2)KZ, W)$$

in the $\mathscr{C}$-manifold case. $K(Z, W)$ is known however by Proposition 2.5, and the proof is complete.

The above development should be compared to that in the Kaehler case [1] and the Sasakian case [8].

We now give a number of geometric results which are consequences of Proposition 2.5.

**Theorem 2.7.** *The sectional curvatures $K(X, Y), X, Y \in \mathscr{L}$, on an $\mathscr{S}$-manifold of constant f-sectional curvature $c < s/4$ satisfy*

$$c \leq K(X, Y) \leq \frac{1}{4}\left(c + \frac{3s}{4}\right)$$

*with the lower limit attained for an f-section. If $c > s/4$,*

$$\frac{1}{4}\left(c + \frac{3s}{4}\right) \leq K(X, Y) \leq c$$

*with the upper limit attained for an f-section. If $c = s/4$, $K(X, Y) = c$.*

*Proof.* Proposition 2.5 gives

$$\begin{aligned} K(X, Y) &= \frac{1}{4}\left(c(1 + 3\cos^2\theta) + \frac{3s}{4}\sin^2\theta\right) \\ &= \frac{1}{4}\left(\left(c + \frac{3s}{4}\right) + 3\left(c - \frac{s}{4}\right)\cos^2\theta\right). \end{aligned}$$

One need only find the maximum and minimum of this with respect to $\theta$ and note that for an $f$-section $\theta = \pi$ to obtain the result.

**Corollary 2.8.** *A Sasakian manifold ($s = 1$) with constant f-sectional curvature equal to $1/4$ has constant curvature.*

*Proof.* By the theorem $s = 1, c = 1/4$ gives $K(X, Y) = 1/4$ for $X, Y \in \mathscr{L}$. Now for any orthonormal pair $\{X, Y\}$ the proof of Theorem 2.6 yields

$$K(X, Y) = \frac{1}{4}\eta_1(X)^2 + \frac{1}{4}\eta_1(Y)^2 + (1 - \eta_1(X)^2 - \eta_1(Y)^2)K(Z, W),$$

$Z, W \in \mathscr{L}$, and hence $K(X, Y) = 1/4$ since $K(Z, W) = 1/4$,

**Theorem 2.9.** *The sectional curvatures $K(X, Y), X, Y \in \mathscr{L}$, on a $\mathscr{C}$-manifold of constant f-sectional curvature c are $(1/4)$-pinched that is $c/4 \leq K(X, Y) \leq c$ for $c > 0$ and $c \leq K(X, Y) \leq c/4$ for $c < 0$. For $c = 0$, the manifold is locally flat (cf. Corollary 1.8).*

*Proof.* By Proposition 2.5, $K(X, Y) = (c/4)(1 + 3\cos^2\theta)$ from which the result follows.

**3.** In this section we start with $M^{2n+s}$ as the bundle space of a principal toroidal bundle over a Kaehler manifold $N^{2n}$; in the case $s = 1$ these are principal circle bundles (see e.g. [2], [7]).

**Theorem 3.1.** *Let $M^{2n+s}$ be the bundle space of a principal toroidal bundle over a Kaehler manifold $N^{2n}$ and let $\gamma = (\eta_1, \cdots, \eta_s)$ be a Lie algebra valued connexion form on $M^{2n+s}$ such that $d\eta_x = \pi^*\Omega, x = 1, \cdots, s$, where $\pi$ is the projection map and $\Omega$ the fundamental 2-form on $N^{2n}$. Then $M^{2n+s}$ is an $\mathcal{S}$-manifold.*

*Proof.* Let $J$ be the almost complex structure tensor and $G$ the Hermitian metric on $N^{2n}$. Then define $f$ and $g$ on $M^{2n+s}$ by

$$fX_m = \tilde{\pi}J\pi_*X_m ,$$
$$g(X, Y) = G(\pi_*X, \pi_*Y) + \sum \eta_x(X)\eta_x(Y) ,$$

where $\tilde{\pi}$ denotes the horizontal lift. Let $\xi_1, \cdots, \xi_s$ be vector fields dual to $\eta_1, \cdots, \eta_s$, i.e., $\eta_x(X) = g(X, \xi_x)$. Then $\eta_x(\xi_y) = \delta_{xy}, f\xi_x = 0, \eta_x \circ f = 0$ are immediate. Now

$$f^2X = \tilde{\pi}J\pi_*\tilde{\pi}J\pi_*X = \tilde{\pi}J^2\pi_*X = -X + \sum \eta_x(X)\xi_x ,$$

from which $f^3 + f = 0$ and we see that $M^{2n+s}$ has an $f$-structure with complemented frames. Further

$$\begin{aligned} g(fX, fY) &= G(J\pi_*X, J\pi_*Y) + \sum \eta_x(\tilde{\pi}J\pi_*X)\eta_x(\tilde{\pi}J\pi_*Y) \\ &= G(\pi_*X, \pi_*Y) = g(X, Y) - \sum \eta_x(X)\eta_x(Y) . \end{aligned}$$

Now $F(X, Y) = g(X, fY) = G(\pi_*X, J\pi_*Y) = \Omega(\pi_*X, \pi_*Y)$, i.e., $F = \pi^*\Omega = d\eta_x$ from which we see that the fundamental 2-form $F$ is closed and that $\eta_1 \wedge \cdots \wedge \eta_s \wedge (d\eta_x)^n \neq 0$. Finally

$$\begin{aligned} [f, f](X, Y) + \sum d\eta_x(X, Y)\xi_x &= f^2[X, Y] + [fX, fY] - f[fX, Y] \\ &\quad - f[X, fY] + \sum d\eta_x(X, Y)\xi_x \\ &= \tilde{\pi}J^2\pi_*[X, Y] + [\tilde{\pi}J\pi_*X, \tilde{\pi}J\pi_*Y] - \tilde{\pi}J\pi_*[\tilde{\pi}J\pi_*X, Y] \\ &\quad - \tilde{\pi}J\pi_*[X, \tilde{\pi}J\pi_*Y] + \sum d\eta_x(X, Y)\xi_x \\ &= \tilde{\pi}J^2[\pi_*X, \pi_*Y] + \tilde{\pi}[J\pi_*X, J\pi_*Y] + \sum \eta_x([\tilde{\pi}J\pi_*X, \tilde{\pi}J\pi_*Y])\xi_x \\ &\quad - \tilde{\pi}J[J\pi_*X, \pi_*Y] - \tilde{\pi}J[\pi_*X, J\pi_*Y] + \sum d\eta_x(X, Y)\xi_x \\ &= -\sum d\eta_x(\tilde{\pi}J\pi_*X, \tilde{\pi}J\pi_*Y)\xi_x + \sum d\eta_x(X, Y)\xi_x \\ &= \sum (-\Omega(J\pi_*X, J\pi_*Y) + \Omega(\pi_*X, \pi_*Y))\xi_x = 0 , \end{aligned}$$

since $[J, J] = 0$ and $\Omega$ is of bidegree $(1, 1)$.

Now let $U$ be a neighborhood on $N^{2n}$ and suppose that $G$ is given by $ds^2 = \sum (\theta^A)^2$, where the $\theta^{A}$'s, $A = 1, \cdots, 2n$ are 1-forms on $U$. Suppose that the Riemannian connexion is given by 1-forms $\theta^A_B$ on $U$ so that the structural equations become

$$d\theta^A = -\theta^A_B \wedge \theta^B ,$$
$$d\theta^A_B = -\theta^A_C \wedge \theta^C_B + \Theta^A_B ,$$

where $\Theta^A_B = \frac{1}{2}S_{ABCD}\theta^C \wedge \theta^D$ and $S_{ABCD}$ is the curvature tensor on $N^{2n}$.

On $U$ write the fundamental 2-form $\Omega = \frac{1}{2}\Omega_{AB}\theta^A \wedge \theta^B$; then we have $d\eta_x = \pi^*(\frac{1}{2}\Omega_{AB}\theta^A \wedge \theta^B)$. Set $\varphi^x = \eta_x$ and $\varphi^A = \pi^*\theta^A$; then $g$ is given by $d\sigma^2 = \sum(\varphi^\alpha)^2$, $\alpha = 1, \cdots, 2n+s$. Using the techniques of Kobayashi [6] we can find the Riemannian connexion on $M^{2n+s}$.

**Proposition 3.2.** *$\varphi^x_y = 0, \varphi^A_x = -\varphi^x_A = -\frac{1}{2}\Omega_{AB}\varphi^B$ and*

$$\varphi^A_B = \pi^*\theta^A_B - \frac{1}{2}\sum_x \Omega_{AB}\varphi^x$$

*define the Riemannian connexion of g on $M^{2n+s}$.*

*Proof.* Let $V$ be an overlapping neighborhood on which $ds^2 = \sum(\bar\theta^A)^2$. Then $\bar\theta^A = e^A_B\theta^B$, $e^A_B \in \mathscr{U}(n)$. A bar above other forms will denote their components defined with respect to $V$. Now

$$\bar\theta^A_B = \sum_{C,D} e^A_C\theta^C_D e^B_D - \sum_C (de^A_C)e^B_C .$$

Let $f^x_\alpha = f^\alpha_x = 0, \alpha \neq x, f^x_x = 1, f^A_B = e^A_B$; then computing we have

$$\sum_{\gamma,\delta} f^x_\gamma\varphi^\gamma_\delta f^y_\delta - \sum_\gamma (df^x_\gamma)f^y_\gamma = 0 = \bar\varphi^x_y ,$$

$$\sum_{\gamma,\delta} f^A_\gamma\varphi^\gamma_\delta f^x_\delta - \sum_\gamma (df^A_\gamma)f^x_\gamma = -\frac{1}{2}\sum_{B,C} e^A_B\Omega_{BC}\varphi^C = -\frac{1}{2}\sum_{B,C,D} e^A_B\Omega_{BC}e^D_C\bar\varphi^D$$
$$= -\frac{1}{2}\bar\Omega_{AD}\bar\varphi^D = \bar\varphi^A_x ,$$

$$\sum_{\gamma,\delta} f^A_\gamma\varphi^\gamma_\delta f^B_\delta - \sum_\gamma (df^A_\gamma)f^B_\gamma = \pi^*\sum_{C,D} e^A_C\theta^C_D e^B_D - \frac{1}{2}\sum_{x,C,D} e^A_C\Omega_{CD}e^B_D\varphi^x$$
$$- \pi^*\sum_C (de^A_C)e^B_C$$
$$= \pi^*\bar\theta^A_B - \frac{1}{2}\sum_x \bar\Omega_{AB}\bar\varphi^x = \bar\varphi^A_B .$$

Hence the $\varphi^\alpha_\beta$ define a connexion on $M^{2n+s}$. To see that it is the Riemannian connexion we compute its torsion.

$$d\varphi^x + \varphi^x_\gamma \wedge \varphi^\gamma = \pi^*\left(\frac{1}{2}\Omega_{AB}\theta^A \wedge \theta^B\right) + \frac{1}{2}\Omega_{AB}\varphi^B \wedge \varphi^A = 0 ,$$

$$d\varphi^A + \varphi^A_\gamma \wedge \varphi^\gamma = \pi^*d\theta^A - \frac{1}{2}\sum_{x,B}\Omega_{AB}\varphi^B \wedge \varphi^x + \left(\pi^*\theta^A_B - \frac{1}{2}\sum_x \Omega_{AB}\varphi^x\right) \wedge \varphi^B$$
$$= \pi^*(d\theta^A + \theta^A_B \wedge \theta^B) = 0 .$$

The curvature form $\Phi^\alpha_\beta$ of this connexion is given by the second structural equation, $d\varphi^\alpha_\beta = -\varphi^\alpha_\gamma \wedge \varphi^\gamma_\beta + \Phi^\alpha_\beta$. Computing $\Phi^A_B$ we have

$$\begin{aligned}
\Phi^A_B &= d\varphi^A_B + \varphi^A_\alpha \wedge \varphi^\alpha_B \\
&= -\pi^*\theta^A_C \wedge \theta^C_B + \pi^*\Theta^A_B - \frac{1}{2}\sum_x (\pi^* d\Omega_{AB}) \wedge \varphi^x \\
&\quad - \frac{1}{2}\sum_x \Omega_{AB} d\varphi^x - \frac{s}{4}\Omega_{AC}\Omega_{BD}\varphi^C \wedge \varphi^D \\
&\quad + \sum_C \left(\pi^*\theta^A_C - \frac{1}{2}\sum_x \Omega_{AC}\varphi^x\right) \wedge \left(\pi^*\theta^C_B - \frac{1}{2}\sum_y \Omega_{CB}\varphi^y\right) \\
&= \pi^*\Theta^A_B - \frac{1}{2}\sum_x (\pi^* d\Omega_{AB}) \wedge \varphi^x + \frac{s}{4}\Omega_{AB}\Omega_{CD}\varphi^D \wedge \varphi^C \\
&\quad - \frac{s}{4}\Omega_{AB}\Omega_{BD}\varphi^C \wedge \varphi^D + \frac{1}{2}\sum_{x,C} \pi^*(\Omega_{AC}\theta^C_B + \Omega_{CB}\theta^C_A) \wedge \varphi^x \\
&\quad + \frac{1}{4}\sum_{x,y,C} \Omega_{AC}\Omega_{CB}\varphi^x \wedge \varphi^y \\
&= \pi^*\Theta^A_B - \frac{s}{4}(\Omega_{AB}\Omega_{CD} + \Omega_{AC}\Omega_{BD})\varphi^C \wedge \varphi^D \\
&\quad + \frac{1}{4}\sum_{x,y,C} \Omega_{AC}\Omega_{CB}\varphi^x \wedge \varphi^y ,
\end{aligned}$$

since $d\Omega_{AB} - \Omega_{AC}\theta^C_B - \Omega_{CB}\theta^C_A = 0$, i.e., $N^{2n}$ is Kaehlerian.

Now write $\Phi^\alpha_\beta = \frac{1}{2}R_{\alpha\beta\gamma\delta}\varphi^\gamma \wedge \varphi^\delta$; then

$$\begin{aligned}
\frac{1}{2}R_{AB\gamma\delta}\varphi^\gamma \wedge \varphi^\delta = \left(\frac{1}{2}S_{ABCD} - \frac{s}{4}(\Omega_{AB}\Omega_{CD} + \Omega_{AC}\Omega_{BD})\right)\varphi^C \wedge \varphi^D \\
+ \frac{1}{4}\sum_{x,y,C} \Omega_{AC}\Omega_{CB}\varphi^x \wedge \varphi^y .
\end{aligned}$$

Skew-symmetrizing gives

$$R_{ABCD} = S_{ABCD} - \frac{s}{4}(2\Omega_{AB}\Omega_{CD} + \Omega_{AC}\Omega_{BD} - \Omega_{AD}\Omega_{BC}) .$$

Suppose now that $N^{2n}$ has constant holomorphic sectional curvature $K$, i.e.,

$$S_{ABCD} = \frac{K}{4}(G_{AD}G_{BC} - G_{AC}G_{BD} + \Omega_{AD}\Omega_{BC} - \Omega_{AC}\Omega_{BD} - 2\Omega_{AB}\Omega_{CD}) .$$

Let $\{X, fX\}$ span an $f$-section on $M^{2n+s}$ with $X$ a unit vector; then the sectional curvature of this section is given by

$$-R_{\alpha\beta\gamma\delta}X^\alpha(fX)^\beta X^\gamma(fX)^\delta = -R_{ABCD}X^A(fX)^B X^C(fX)^D$$
$$= -\frac{K}{4}(G_{AD}G_{BC} - G_{AC}G_{BD})X^A(fX)^B X^C(fX)^D$$
$$+ \left(\frac{s}{4} - \frac{K}{4}\right)(\Omega_{AD}\Omega_{BC} - \Omega_{AC}\Omega_{BD} - 2\Omega_{AB}\Omega_{CD})X^A(fX)^B X^C(fX)^D$$
$$= \frac{K}{4} + \frac{3K}{4} - \frac{3s}{4} = K - \frac{3s}{4}.$$

Hence we have the following theorem.

**Theorem 3.3.** *Let $M^{2n+s}$ be a principal toroidal bundle over a Kaehler manifold $N^{2n}$ as in Theorem 3.1. If $N^{2n}$ has constant holomorphic sectional curvature $K$, then the $\mathcal{S}$-manifold $M^{2n+s}$ has constant f-sectional curvature equal to $K - 3s/4$.*

Inequalities for the sectional curvature of other horizontal sections may be derived from Theorem 2.7.

**4.** It is well-known that the canonical example of a Sasakian manifold, the odd-dimensional sphere $S^{2n+1}$, is a circle bundle over complex projective space $PC^n$ by the Hopf-fibration. Let $\pi' : S^{2n+1} \to PC^n$ denote the Hopf-fibration; then using the diagonal map $\Delta$ we define a principal toroidal bundle over $PC^n$ by the following diagram

$$\begin{array}{ccc} H^{2n+s} & \xrightarrow{\hat{\Delta}} & S^{2n+1} \times \cdots \times S^{2n+1} \\ \downarrow & & \downarrow{\scriptstyle \pi' \times \cdots \times \pi'} \\ PC^n & \xrightarrow{\Delta} & PC^n \times \cdots \times PC^n \end{array}$$

that is, $H^{2n+s} = \{(p_1, \cdots, p_s) \in S^{2n+1} \times \cdots \times S^{2n+1} | \pi'(p_1) = \cdots = \pi'(p_s)\}$.

Now let $\eta'_x$ be the contact form on $S^{2n+1}_x$ and define $\eta_x$ on $H^{2n+s}$ by $\eta_x = \hat{\Delta}^*|_{S^{2n+1}_x}\eta'_x \equiv \hat{\Delta}^*_x\eta'_x$. Then

$$d\eta_x = d\hat{\Delta}^*_x\eta'_x = \hat{\Delta}^*_x d\eta'_x = \hat{\Delta}^*_x\pi'^*_x\Omega_x = \pi^*\Delta^*_x\Omega_x = \pi^*\Omega ,$$

where $\Omega_x$ is the fundamental 2-form on $PC^n_x$ and $\Omega$ that on $PC^n$. Further $\gamma = (\eta_1, \cdots, \eta_s)$ is equivariant and fibre preserving, hence by Theorem 3.1 the space $H^{2n+s}$ is an $\mathcal{S}$-manifold.

Recall that $PC^n$ has constant holomorphic sectional curvature $K = 1$ (Fubini-Study metric) and that $S^{2n+1}$ (as a Sasakian manifold with the constant curvature metric) has constant curvature $1/4$. From Theorem 3.3 we obtain the following result.

**Theorem 4.1.** *$H^{2n+s}$ has constant f-sectional curvature $1 - 3s/4$.*

Analogous to $PC^n$ being $(1/4)$-pinched $(1/4 \leq K(X, Y) \leq 1)$ and $S^{2n+1}$ having constant curvature $1/4$, from Theorems 2.7 and 4.1 we have

**Theorem 4.2.** *Let $X, Y \in \mathcal{L}$ on $H^{2n+s}$, $s \geq 2$. Then*

$$1-\frac{3s}{4}\leq K(X,Y)\leq\frac{1}{4}\,.$$

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